Math 656 • March 10, 2011

Midterm Examination

This is a closed-book exam; neither notes nor calculators are allowed. Explain your work

Note: points add up to 108. You only need 100 points.

1) (14pts) Derive the expression for $\sinh^{-1} z$ (arcsinh z) using the definition of sinh w in terms of exponentials, and use it to find **all** values of $\sinh^{-1}(2i)$. Plot these values as points in the complex plane. Make sure your points agree with the period of the hyperbolic sine function.

2) (15pts) Show that images of vertical lines under transformation $w = \cos z$ are hyperbolic $\left(\frac{u^2}{a^2} - \frac{v^2}{b^2} = \pm 1\right)$. What shape are images of horizontal lines? What is the angle between these two sets of curves?

3) (16pts) Is the function $f(z) = z/\bar{z}$ continuous for all z? Is it differentiable anywhere? Is it analytic anywhere? Prove your answers directly (using limits), and verify your answer about analyticity using Cauchy-Riemann equations.

4) (16pts) Find *all* branch points of $\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$, and find the branch cut for *any* branch choice for this function (hint: this is simpler than examples we solved in class). How much does the function jump across the branch cut(s)?

5) (16pts) Use parametrization to show that the following integral is zero over any circle around the origin:

$$\oint_{|z|=R} \left(\frac{1}{\bar{z}} + z\right) dz$$

Does it follow that the integrand has an anti-derivative everywhere in the domain C/{0}? Does it follow that the integrand is analytic in this domain? Explain your answers.

6) (16pts) Without resorting to parametrization, calculate $\int_C \frac{z\,dz}{z^2 + i}$ along two different contours:

a) C = any contour from z = 1 to z = –1 not containing singularities of integrand

b) C = circle of radius 2 around the origin in the positive direction

7) (15pts) Parametrize the integral $\oint_{|z|=1} \frac{e^{\alpha z}}{z} dz$ (where α is a real constant) and use the Cauchy Integral Formula to show that $\int_0^{\pi} e^{\alpha\cos\theta}\cos(\alpha\sin\theta)\,d\theta = \pi$.